

## Periodic orbits analysis of the form factor: from ballistic to diffusive systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1996 J. Phys. A: Math. Gen. 29 2013

(<http://iopscience.iop.org/0305-4470/29/9/016>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.71

The article was downloaded on 02/06/2010 at 04:11

Please note that [terms and conditions apply](#).

# Periodic orbits analysis of the form factor: from ballistic to diffusive systems

Oded Agam<sup>†</sup> and Shmuel Fishman<sup>‡</sup>

Department of Physics, Technion, Haifa 32000, Israel

Received 29 August 1995

**Abstract.** The energy level correlator  $K(s)$  and the form factor  $\hat{K}(t)$  are calculated for a hypercubic billiard with small hyperspheres placed at random in its interior. Various regimes, characterized by the elastic mean free path  $l$ , resulting from the scattering on impurities, are identified. The analysis extends from the ballistic regime, where  $l$  is much larger than the size of the system, via intermediate regimes, to the diffusive regime, where  $l$  is much smaller than its size. Semiclassical expressions for the density of states of chaotic and integrable systems in terms of classical periodic orbits are used. The diagonal approximation for  $\hat{K}(t)$  is made for short times, while non-perturbative methods are used for long times. The analysis makes use of analytic properties of classical dynamical zeta function associated with the Perron–Frobenius operator. The general features are relevant for mesoscopic systems.

## 1. Introduction

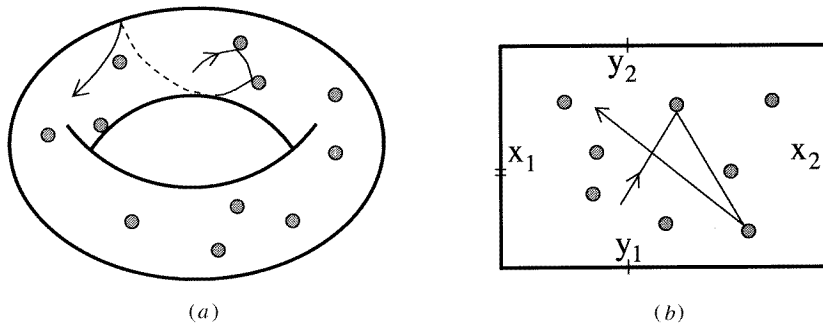
The fingerprints of the underlying classical chaotic dynamics on the quantal behaviour of the corresponding system are explored in the field of ‘quantum chaos’ [1–3]. In particular it was found that the level statistics of a variety of chaotic systems are described by random matrix theory (RMT) [4] over a certain energy domain [1, 5]. Examples of such systems are the Sinai and the stadium billiards. Beyond this energy domain there are deviations from the RMT universal behaviour. The reason is that long energy scales correspond to short time scales over which the behaviour of the system is not universal. For ballistic chaotic systems this time scale is of the order of the period of the shortest periodic orbits [6]. For disordered metallic grains it is the Thouless time, namely the time required for the particle to diffuse across the system [7]. The corresponding energy scale is the Thouless energy. On shorter energy scales universality applies, while on longer energy scales deviations from universality are expected.

In contrast with chaotic systems, the spectral statistics of generic integrable systems, such as the rectangular billiard, is Poissonian [8]. Since symmetry is crucial for such systems, it was for a long time considered impractical to realize them in the mesoscopic domain.

The advance in fabrication techniques enables us to manufacture mesoscopic systems that are relatively clean, namely their elastic mean free path can be larger than the size of the system. Such systems were prepared with boundaries of different shapes and used to explore the difference between transport in chaotic and in integrable systems [9–11].

<sup>†</sup> Present address: NECL, 4 Independence Way, Princeton, NJ 08540, USA.

<sup>‡</sup> Member of the Minerva Center for Nonlinear Physics of Complex Systems.



**Figure 1.** Torus with impurities and the corresponding billiard with points on opposite sides ( $x_1$  and  $x_2$ ,  $y_1$  and  $y_2$ ) identical.

Theoretical explorations were performed in parallel with the experimental investigations [12–14]. Several experimental results [15, 16] on relatively clean systems were explained in the framework of the semiclassical methods that were developed in the field of ‘quantum chaos’ [17–20]. One can prepare nanostructure devices so that their boundaries induce integrable classical dynamics if there is no disorder, e.g. when the boundary is a circle or a rectangle. Since the level statistics of generic integrable systems is Poissonian [8], one expects that the results of RMT cannot be applied directly for such systems. Yet, mesoscopic systems, even if prepared with a very high degree of cleanliness, inevitably contain some impurities. This motivates the study of a model in which the amount of disorder can be controlled, say by fixing the number of impurities, and with the property that without disorder the classical dynamics is integrable. It is worthwhile mentioning here that such a system was realized experimentally using microwave cavities [21].

In earlier work, Altland and Gefen (AG) explored a related model where pointlike impurities were introduced into an integrable system [22, 23]. Both the diffusive regime, where the elastic mean free path is much smaller than the size of the system, and the clean regime, where it is much larger than the system were explored. In their investigation, AG calculated the averaged correlation function between energy levels. They studied in detail the difference between energy averaging and ensemble averaging over realizations of impurities. In their calculations diagrammatic techniques were used.

In the present paper a  $D$ -dimensional Lorentz gas [24] in a finite volume will be studied. This is a  $D$ -dimensional cubic billiard doped with rigid spherical impurities. The boundary conditions are assumed to be periodic, so that the geometry is of a  $D$ -dimensional torus. Such a system is illustrated in figure 1 for  $D = 2$ . This model is a prototype system for integrable systems (where the motion in phase space is on tori) doped with impurities that scatter particles only within some finite range of their centre. The trajectories that are not scattered by impurities are identical to those of the corresponding integrable system. The trajectories that are scattered by impurities may be chaotic, and are such if the impurities are rigid spheres. The calculations will be done in the framework of the semiclassical approximation. For the validity of this approximation it will be assumed that the energy is sufficiently high so that the wavelength of the particles is much smaller than the radius of the spheres. It will be assumed also that all the spheres have the same radius, that is much smaller than the size of the system. This model is therefore somewhat different from the model studied by AG [23]. The classical motion for this prototype system resembles the motion of a large variety of classical systems in angle-action variables. This similarity

also holds in the framework of the semiclassical approximation [25]. The calculations in this paper will be performed in the framework of the semiclassical approximation, hence, effects of diffraction, tunnelling and ghost orbits that were found important in various studies [26, 27], will not be taken into account here and will be left for further studies.

It is proper to mention here that even with only one impurity our model system is completely chaotic. In fact it is the  $D$ -dimensional Sinai billiard which was proved to be a  $K$  system. Thus, over sufficiently long time scales in which the motion of the particle is ergodic, the properties of this system are expected to be the universal ones described by RMT. Here we shall be mainly interested in time scales where the deviations from universality are important. We thus also consider short time scales over which the classical dynamics may still be far from the ergodic limit.

The specific quantity that will be explored in detail is the dimensionless density–density correlator. It is defined to be

$$K(s) = \Delta^2 \langle \rho(\epsilon) \rho(\epsilon + s\Delta) \rangle_\epsilon - 1 \quad (1.1)$$

where  $\rho(\epsilon)$  is the density of states at energy  $\epsilon$ ,  $\Delta$  is the mean level spacing, and  $\langle \dots \rangle_\epsilon$  represents an averaging over some interval of the energy  $\epsilon$  for a specific realization of disorder. An ensemble average over realizations of disorder is denoted hereafter by  $\langle \dots \rangle$ . In what follows the ensemble averaged correlator,  $\langle K(s) \rangle$ , will be calculated. A related function which is usually used in this context is the dimensionless form factor which is the Fourier transform of  $K(s)$ . Denoting it by  $\hat{K}(t)$  it is given by

$$\hat{K}(t) = \int ds K(s) e^{-ist}. \quad (1.2)$$

The semiclassical formula of density of states  $\rho(\epsilon)$ , in our system, may be represented as a sum of three terms:

$$\rho(\epsilon) = \rho_0(\epsilon) + \rho_{po}^{(i)}(\epsilon) + \rho_{po}^{(c)}(\epsilon). \quad (1.3)$$

Here  $\rho_0(\epsilon) = 1/\Delta$  is the Weyl term. To the leading order in one over Planck's constant  $h$ , it takes the form

$$\rho_0(\epsilon) = \frac{1}{h^D} \int d^D r d^D p \delta[\epsilon - \mathcal{H}(\mathbf{r}, \mathbf{p})] \quad (1.4)$$

where  $\mathbf{r}$  and  $\mathbf{p}$  are the coordinate and the momentum vectors respectively, and  $\mathcal{H}(\mathbf{r}, \mathbf{p})$  is the Hamiltonian. The last two terms in (1.3) are contributions of the periodic orbits. The first term  $\rho_{po}^{(i)}$  is the contribution from orbits that are not scattered by impurities. This contribution will be calculated in section 2 following the calculation by Berry and Tabor [25]. A similar contribution was found to be important for spectral statistics of some chaotic systems, such the stadium billiard [28]. The second term  $\rho_{po}^{(c)}$  results from the scattered orbits and is given by the Gutzwiller trace formula [29]. Each of the last two terms in (1.3) takes the form

$$\rho_{po}(\epsilon) = \sum_{\mu} A_{\mu} e^{iS_{\mu}} / \hbar \quad (1.5)$$

where  $A_{\mu}$  is the amplitude of the  $\mu$ th orbit, while  $S_{\mu}$  is its action. The amplitudes  $A_{\mu}$  for the integrable and chaotic contributions were calculated in [25, 29], and will be presented in sections 2 and 3.

Inserting the semiclassical expression for the density of states (1.3) in (1.1), and keeping only the diagonal part of the double sum over periodic orbits, leads to the diagonal approximation [30] of the form factor

$$\hat{K}_P(t) = \hat{K}_P^{(i)}(t) + \hat{K}_P^{(c)}(t). \quad (1.6)$$

Here  $K_P^{(i)}$  and  $K_P^{(c)}$  are the contributions of periodic orbits that are not scattered and that are scattered by impurities, respectively. In this approximation the products of terms belonging to different periodic orbits, which are not related by symmetry, are assumed to be averaged out to zero. For metallic grains it was found to give results identical to those of diagrammatic perturbation theory [31]. The subscript  $P$  is therefore used here to denote the perturbative results.

Throughout this paper dimensionless quantities will be used. In particular, the length of the billiard and the mass of the particle are chosen to be equal to unity; energy is measured in units of the mean level spacing ( $s = \epsilon/\Delta$ ) and time is measured in units of  $\hbar/\Delta$ . In these units the Heisenberg time is  $\tau_H = h\rho_0 = 2\pi$ .

Having the semiclassical density–density correlator, which was obtained in the diagonal approximation, one can go beyond this approximation.  $K(s)$  may be represented as a sum of two terms [32, 33]:

$$K(s) = K_P(s) + K_{osc}(s). \quad (1.7)$$

The first,  $K_P(s)$ , is the smooth term given by the diagonal approximation discussed above or calculated by diagrammatic perturbation theory. It can be represented in the form

$$K_P(s) = -\frac{\beta}{4\pi^2} \frac{\partial^2}{\partial s^2} \ln[\mathcal{D}(s)] \quad (1.8)$$

where  $\beta$  equals two for systems which are invariant with respect to time reversal and unity for the unitary ensemble.  $\mathcal{D}(s)$  is a classical spectral determinant associated with the decaying modes of a disturbance in the density of particles in the system. For instance, in the diffusive regime it is given by

$$\mathcal{D}(s) = \prod_{\mu} B_{\mu}(s^2 + \epsilon_{\mu}^2)^{-1} \quad (1.9)$$

where  $\epsilon_{\mu}$  are the eigenvalues (in units of  $\Delta$ ) of the diffusion equation, and  $B_{\mu}$  are regularization factors introduced to make the product converge. The spectral determinant  $\mathcal{D}(s)$  is normalized such that

$$\lim_{s \rightarrow 0} s^2 \mathcal{D}(s) = 1. \quad (1.10)$$

The second term,  $K_{osc}$ , cannot be calculated by usual perturbation theory. It was obtained for hyperbolic systems in the limit of large  $s$  retaining the non-perturbative terms of the correlator [32]. It turns out [32, 33] that  $K_{osc}$  is also governed by the same classical spectral determinant, namely for the unitary (u) and orthogonal (o) cases it has the forms:

$$K_{osc}^u(s) = \frac{\cos(2\pi s)}{2\pi^2} \mathcal{D}(s) \quad \text{and} \quad K_{osc}^o(s) = -\frac{\cos(2\pi s)}{2\pi^4} \mathcal{D}^2(s). \quad (1.11)$$

In this paper periodic orbit theory will be used in order to calculate the perturbative part of the density correlator  $K_P(s)$  as outlined in the discussion leading to (1.6). Then by identifying the spectral determinant  $\mathcal{D}(s)$  we shall be able to extend the result to include also the non-perturbative part  $K_{osc}(s)$  and in this way we will be able to go beyond the diagonal approximation.

The outline of the paper is as follows. The contribution of orbits that are not scattered by impurities to the form factor,  $\hat{K}^{(i)}$ , will be calculated in section 2, while  $\hat{K}^{(c)}$  the contribution of orbits that are scattered by impurities will be calculated in section 3. The spectral determinant  $\mathcal{D}(s)$  will be calculated in section 4. Calculations of the density–density correlator  $K(s)$  and of  $\hat{K}(t)$  for  $D = 2$  degrees of freedom will be presented in section 5. Finally the results will be discussed in section 6.

## 2. The contribution of trajectories that are not scattered by impurities

In this section the contribution of trajectories that are not scattered by impurities is calculated. The contribution of each trajectory of this type is identical to the corresponding contribution for an integrable system. The periodic orbits contribution to the density of states for integrable systems was calculated by Berry and Tabor (BT) [25]. Using BT equation (2),

$$\rho(s) = \Re \frac{1}{\pi} \int_0^\infty dt e^{ist} \int d^D r U(\mathbf{r}, \mathbf{r}; t) \quad (2.1)$$

where  $U$  is the propagator. Their calculation exploits the fact, that when expressed in angle-action variables the trace of the semiclassical propagator can be written in the form (BT equation (11))

$$\int d^D r U_{sc}(\mathbf{r}, \mathbf{r}; t) = \sum_M \int_0^{2\pi} d^D \Theta U_{sc}(\Theta + 2\pi \mathbf{M}, \Theta; t) \quad (2.2)$$

where the vector  $\mathbf{M}$  describes an orbit that after time  $t$  returns to its initial point on the torus, after making  $M_1$  circuits  $\theta_1$ ,  $M_2$  circuits of  $\theta_2$ , etc. All  $M_i$  are non-negative. Clearly  $\mathbf{M}$  labels the topology of the periodic orbits. Each term in the sum (2.2) is found semiclassically to be independent of  $\Theta$ , therefore the  $\Theta$  integration contributes a factor  $(2\pi)^D$ . Doing the time integral in (2.1) yields the following contribution of the periodic orbits to the density of states (BT equation (17)):

$$\rho_{po}^{(integrable)} = 2\Re \sum_M' A_M \exp \left\{ i 2\pi \mathbf{M} \left( \frac{\mathbf{I}_M}{\hbar} - \frac{\boldsymbol{\alpha}_M}{4} \right) \right\} \quad (2.3)$$

where  $\mathbf{I}_M$  and  $\boldsymbol{\alpha}_M$  are vectors of actions and Maslov indices of the orbit of topology  $\mathbf{M}$ . The prime over the sum denotes exclusion of terms with negative  $M_i$  as well as of the term  $\mathbf{M} = 0$ . The amplitudes  $A_M$  were calculated by Berry and Tabor [25] (see equation 18 and section 4 there).

For a system with impurities, that are here assumed to be spheres, the angles corresponding to orbits that are scattered by impurities should be excluded from the  $\Theta$  integral in (2.2). The resulting contribution of the periodic orbits that are not scattered by impurities is

$$\rho_{po}^{(i)} = 2\Re \sum_M' f_M A_M \exp \left\{ i 2\pi \mathbf{M} \left( \frac{\mathbf{I}_M}{\hbar} - \frac{\boldsymbol{\alpha}_M}{4} \right) \right\} \quad (2.4)$$

where  $f_M$  is the fraction of the trajectories with the topology  $\mathbf{M}$  that are not scattered by impurities. It is a random variable that depends on the realization of the configuration of impurities.

The resulting contribution to the perturbative (namely diagonal) part of the form factor (1.2) is

$$\hat{K}_P^{(i)}(t) = 2\pi\beta \sum_M' \delta(t - t_M) |A_M|^2 f_M^2 \quad (2.5)$$

where  $t_M$  is the period of an orbit of the topology  $\mathbf{M}$  while  $\beta = 2$  if the system is time reversible and  $\beta = 1$  if it is not. Notice that time here is dimensionless (measured in units of  $\hbar/\Delta$ ). This contribution, when averaged over realizations, is

$$\langle \hat{K}_P^{(i)}(t) \rangle = 2\pi\beta \sum_M' \delta(t - t_M) |A_M|^2 \langle f_M^2 \rangle. \quad (2.6)$$

In what follows  $\langle f_M^2 \rangle$  will be estimated. It will turn out to be a smooth function of the length of the orbit, or its period. It will be denoted as

$$\varphi(t) = \langle f_M^2 \rangle \quad (2.7)$$

where  $t = t_M$ . The Hannay–Ozorio de Almeida sum rule [34] implies

$$\sum_M' \delta(t - t_M) |A_M|^2 = \frac{1}{2\pi}. \quad (2.8)$$

The resulting expression for the contribution to the dimensionless form factor is

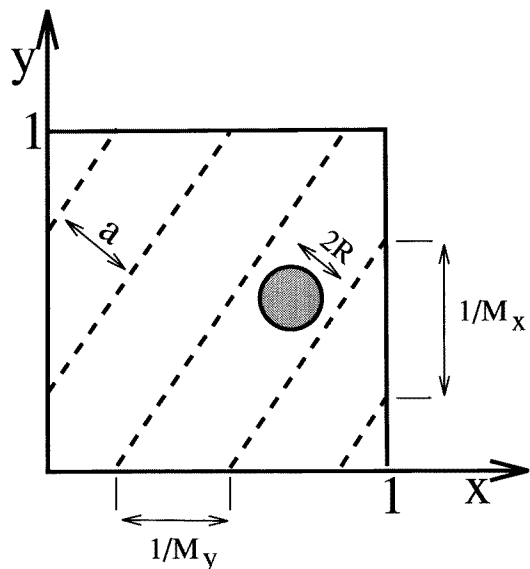
$$\langle \hat{K}_P^{(i)}(t) \rangle = \beta \varphi(t). \quad (2.9)$$

The function  $\varphi(t)$  will be estimated in what follows. First it is estimated for a system with  $D = 2$  degrees of freedom. Then it is argued heuristically that the result holds for any  $D$  (see equation 2.24 and the discussion that follows). It is instructive to assume first that there is just one circular impurity of radius  $R$ . This is just the Sinai billiard problem. Consider  $M = (M_x, M_y)$ , depicted in figure 2. A torus geometry is assumed with opposite sides identified. The units of length are normalized so that the sides of the square are of unit length and the corresponding angle variables are  $\theta_x = 2\pi x$  and  $\theta_y = 2\pi y$ . The distance between neighbouring segments of an orbit of the topology  $M$  is

$$a = \frac{1}{\sqrt{M_x^2 + M_y^2}}. \quad (2.10)$$

If  $2R > a$  all orbits of this topology are scattered by the circle. For  $2R < a$  the fraction of the orbits that are not scattered is  $(a - 2R)/a$ . If we choose a specific orbit, the probability that it is not scattered by the circle is

$$P_1 = 1 - \frac{2R}{a}. \quad (2.11)$$



**Figure 2.** Example of a periodic orbit with topology  $M = (2, 3)$  (broken line).

Now if  $N$  identical small circles of radius  $R \ll 1$  are placed in the square at random, the probability that an orbit with a given  $\Theta$  is not scattered by any circle is

$$P_N \approx \left(1 - \frac{2R}{a}\right)^N \quad (2.12)$$

since the circles are small and their overlap probability is of the order  $R^2$ . If the limit  $R \rightarrow 0$  and  $N \rightarrow \infty$  is taken so that  $\tilde{R} = 2RN$  is constant, the probability that a specific orbit is scattered is

$$P = e^{-\tilde{R}/a}. \quad (2.13)$$

Note that in this limit the area that is covered by the circles vanishes. In this derivation it was assumed that  $2R \ll a$ . This assumption will be made throughout this section. At the end of this section it will be justified for periodic orbits that are of interest for this system.

We are interested in the random variable  $f_M$  that is the fraction of angles for which the trajectories are not scattered by circles. In the limit of small circles this is identical to the following problem, known as the covering problem [35–37]. Given a circle of length  $a$ ,  $N$  arcs of length  $2R$  are dropped at random on this circle with uniform probability. The uncovered part is the random variable  $f_M$ . In the limit  $R \rightarrow 0$ ,  $N \rightarrow \infty$  but  $\tilde{R}$  fixed, the first two moments are [36] (equations (3.13) and (3.11) there)

$$\langle f_M \rangle = e^{-\tilde{R}/a} \quad (2.14)$$

and

$$\langle f_M^2 \rangle = e^{-2\tilde{R}/a}. \quad (2.15)$$

This result can be easily understood since

$$f_M = \frac{1}{a} \int_0^a dx \chi(x) \quad (2.16)$$

where  $\chi(x)$  is the characteristic variable taking the value 1 for an uncovered point  $x$  and the value 0 for a covered one. Its average is  $\langle \chi(x) \rangle = P_N$ , and (2.14) follows immediately. The second moment is

$$\langle f_M^2 \rangle = \frac{1}{a^2} \int_0^a \int_0^a dx dy \langle \chi(x) \chi(y) \rangle. \quad (2.17)$$

The variables  $\chi(x)$  and  $\chi(y)$  are not independent, but the distance  $|x - y|$  over which they are correlated is of order  $R$  and therefore vanishes in the limit discussed here. Consequently  $\langle \chi(x) \chi(y) \rangle \approx \langle \chi(x) \rangle \langle \chi(y) \rangle$ , leading to (2.15).

In order to write (2.15) in more familiar terms note that

$$\frac{\tilde{R}}{a} = 2RN \sqrt{M_x^2 + M_y^2} = \frac{L_M}{l} \quad (2.18)$$

where  $l = 1/2RN$  is the classical elastic mean free path ( $2R$  is the classical cross section), while  $L_M$  is the length of the orbit of the topology  $M$ . Consequently

$$\langle f_M^2 \rangle = e^{-2L_M/l} \quad (2.19)$$

and returning to definition (2.7)

$$\varphi(t) = \langle f_M^2 \rangle = e^{-2t/\tau} \quad (2.20)$$

where  $\tau = l/v$  and  $t = L_M/v$ , while  $v$  is the velocity.



In this derivation we took the limit  $R \rightarrow 0$ , that is valid for

$$2R \ll a = \frac{1}{L_M} \quad (2.21)$$

or

$$L_M \ll lN. \quad (2.22)$$

For very long orbits where these inequalities are not satisfied,

$$\langle f_M^2 \rangle \approx \left(1 - 4\frac{R}{a}\right)^N \quad (2.23)$$

for  $N \gg 1$  (see [36], equation (3.11)). It reduces to (2.15) if (2.21) holds, but in general the right-hand side of (2.15) and (2.19) is an upper bound on (2.23). But if (2.22) is violated,  $\langle f_M^2 \rangle$  is bounded by a term of the order of  $e^{-2N}$ , that is negligible for large  $N$ . Therefore (2.15), leading to (2.19) and (2.20), was used in this section.

Now it will be argued that (2.19) and (2.20) hold for any dimension. From the definition of the elastic mean free path (assuming it is meaningful), the probability that an orbit of topology  $M$  (length  $L_M$ ) crossing the point  $\Theta$  on the torus is not scattered by any impurity is  $e^{-L_M/l}$ . The characteristic variable defined following (2.16) is now extended to take  $\chi(\Theta) = 1$  if the orbit passing through  $\Theta$  is not scattered by an impurity and the values  $\chi(\Theta) = 0$  if it is scattered. Again,

$$f_M = \frac{1}{(2\pi)^D} \int_0^{2\pi} \cdots \int_0^{2\pi} d^D \Theta \chi(\Theta). \quad (2.24)$$

The equation corresponding to (2.16) is now

$$\langle f_M^2 \rangle = \frac{1}{(2\pi)^{2D}} \int_0^{2\pi} \cdots \int_0^{2\pi} d^D \Theta d^D \Theta' \langle \chi(\Theta) \chi(\Theta') \rangle. \quad (2.25)$$

Since the correlations are only over small distances, one can assume  $\langle \chi(\Theta) \chi(\Theta') \rangle \approx \langle \chi(\Theta) \rangle \langle \chi(\Theta') \rangle$ , leading to (2.19) and (2.20) for any dimension. Note that this argument is heuristic, while the estimate for  $D = 2$  is rigorous.

The resulting contribution of the orbits that are not scattered by impurities to the form factor given by (2.9) is

$$\langle \hat{K}_P^{(i)}(t) \rangle = \beta e^{-2t/\tau} \quad (2.26)$$

where  $t > 0$ . The extension to  $t < 0$  is obtained, replacing  $t$  by its absolute value. The density–density correlator, i.e. the inverse Fourier transform of  $\hat{K}_P(t)$  is

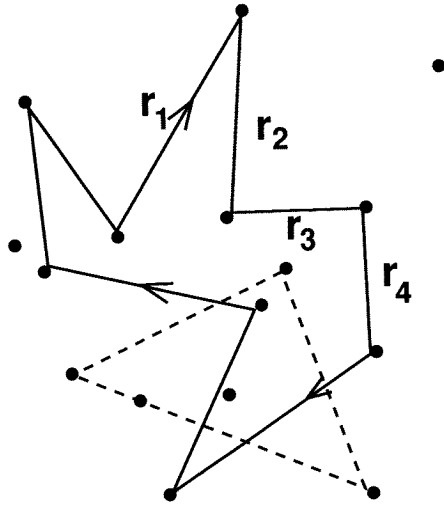
$$\langle K_P^{(i)}(s) \rangle = \frac{2\beta}{\pi} \frac{\tau}{4 + (s\tau)^2}. \quad (2.27)$$

This corresponds to  $K'$  in the terminology of Altland and Gefen, see equation (47) and (52) in [23].

### 3. The contribution of orbits which are scattered by impurities

In this section, the perturbative part of the density correlator,  $K_P^{(c)}(s)$ , associated with the periodic orbits which are scattered from impurities will be calculated. This part emerges from the contribution of isolated orbits to the (dimensionless) density of states, that is given by Gutzwiller's trace formula:

$$\rho_{po}(s) = \Re \frac{1}{\pi} \sum_{\mu} t_{\mu} \sum_r \frac{e^{(i/\hbar)S_{\mu}(\epsilon)r - iv_{\mu}r}}{|\det(\mathcal{M}_{\mu}^r - I)|^{1/2}} \quad (3.1)$$



**Figure 3.** An illustration of the construction of orbits that are scattered by impurities. The full circles represent the impurities, and the full lines are a periodic orbit which is scattered from them. The broken line illustrates a forbidden orbit.

where  $\mu$  labels a primitive orbit that is characterized by a period  $t_\mu$ , action  $S_\mu(\epsilon)$ , and Maslov phase  $\nu_\mu$ .  $r$  stands for the number of the repetitions of this orbit, and  $\mathcal{M}_\mu$  is the monodromy matrix associated with the linearized dynamics on the Poincaré section perpendicular to the orbit.

Our working assumption is that the periodic orbits can be constructed from sets of segments connecting the impurities, as illustrated for example by the full curve in figure 3. Thus among any ordered set of impurities one can construct a periodic orbit unless a segment between two impurities hits another impurity. This situation is illustrated by the broken curve in figure 3. It is thus assumed that a particle can be scattered from an impurity in any direction. This approximation is valid as long as the radius of the spherical impurities is small compared to the average distance between nearby scatterers.

We shall construct  $K_p^{(c)}(s)$  as a sum

$$K_p^{(c)}(s) = \sum_{j=1}^{\infty} K_j(s) \tag{3.2}$$

where  $K_j(s)$  denotes the contribution from orbits which are scattered  $j$  times from impurities. Within the diagonal approximation the semiclassical formula for  $K_j(s)$  is given as a sum over orbits,

$$K_j(s) = \frac{-\beta}{2\pi^2} \frac{\partial^2}{\partial s^2} \Re \sum_{\mu(j)} |\det(\mathcal{M}_\mu - \mathbf{1})|^{-1} e^{ist_\mu} \tag{3.3}$$

where  $\mu = \mu(j)$  denotes all orbits which are scattered from  $j$  impurities,  $t_\mu$  is the period of the  $\mu$ th orbit,  $\mathcal{M}_\mu$  is the corresponding monodromy matrix which describes the linearized dynamics in the vicinity of the orbit and  $\mathbf{1}$  is the unit matrix. In the above formula we have neglected the contribution from orbits which are not primitive orbits, namely those which are repetitions of shorter ones. It is justified due to the exponential proliferation in the number of periodic orbits as their length increases. Formula (3.3) corresponds to a specific realization of the configuration of impurities.

Let us start by estimating the number of periodic orbits which are scattered from  $j$  impurities. We denote by the index  $i$  the  $i$ th segment of such an orbit. We can start the construction of an orbit from any of the  $N$  impurities which exist in the system. The

number of possibilities in going to the next impurity near  $r_1$  in the vicinity  $d^D r_1$  is given by  $N r_1^{D-1} dr_1 d\Omega_1 e^{-r_1/l}$  where  $N$  is the number of the impurities which in our units is also their density;  $d\Omega_1$  is an angle element of a  $D$ -dimensional hypersphere, and  $e^{-r_1/l}$  is the probability that the orbit does not hit any impurity on the way. Here  $l$  is the classical elastic mean free path. Thus the total number of orbits in the vicinity of the sequence of segments  $(r_1, r_2, \dots, r_j)$  is

$$N^j \prod_{i=1}^j e^{-r_i/l} r_i^{D-1} dr_i d\Omega_i. \quad (3.4)$$

Notice that here we still do not assume that the orbit is periodic.

Our next step is to estimate the amplitude,  $|\det(\mathcal{M}_\mu - \mathbf{1})|^{-1}$  of an orbit which undergoes the sequence of scattering represented by  $\mu = (r_1, r_2, \dots, r_j)$ . Denoting by  $\Lambda_v^+$  (where  $v = 1, 2, \dots, D-1$ ) the absolute values of the eigenvalues of  $\mathcal{M}_\mu$  (that are larger than one), one has

$$|\det(\mathcal{M}_\mu - \mathbf{1})| \simeq \prod_{v=1}^{D-1} \Lambda_v^+ \quad (3.5)$$

where it is assumed that the orbits are very unstable. This assumption will turn out to hold in the limit where the spheres are small. In what follows we shall estimate the eigenvalues of the monodromy matrices.

The monodromy matrix has a multiplicative property, namely, it can be constructed as a product of matrices each one associated with a different segment of the orbit,

$$\mathcal{M}_\mu = \mathcal{M}_\mu^{(j)} \dots \mathcal{M}_\mu^{(2)} \mathcal{M}_\mu^{(1)} \quad (3.6)$$

where  $\mathcal{M}_\mu^{(i)}$  corresponds to the segment of the orbit that is composed of free motion along  $r_i$  and a scattering from the impurity. The matrix  $\mathcal{M}_\mu^{(i)}$  is thus given by the product  $\mathcal{M}_{\text{collision}} \mathcal{M}_{\text{freeflight}}$ , where  $\mathcal{M}_{\text{collision}}$  and  $\mathcal{M}_{\text{freeflight}}$  correspond to the collision with the impurity and the free motion parts of the monodromy matrix respectively. To express these matrices it is convenient to use the following coordinate system shown in figure 4:  $r_{\parallel}$  is the coordinate along the orbit;  $r_{\perp}$  orthogonal to  $r_{\parallel}$  and is in the plane defined by the two segments of the orbit before and after the scattering, and  $r_b$  represents the set of remaining coordinates. We shall denote by  $p_{\perp}$  and  $p_b$  the momenta conjugate to  $r_{\perp}$  and  $r_b$ , respectively.

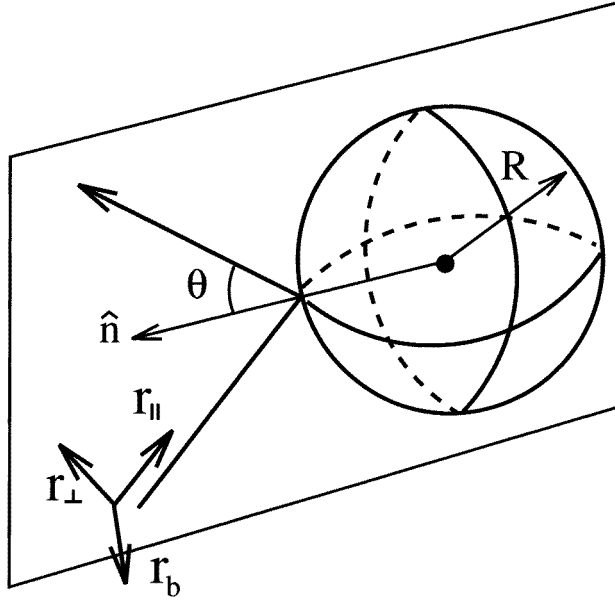
The monodromy matrix is defined on the linear subspace perpendicular to  $r_{\parallel}$  in phase space. It is convenient to choose the vectors in this space to be of the form  $(r_{\perp}, p_{\perp}, r_b, p_b)^T$ . In this representation a free flight along a distance  $r$  corresponds to a monodromy matrix:

$$\mathcal{M}_{\text{freeflight}} = \begin{pmatrix} 1 & r & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & \mathbf{1} & \frac{r}{p} \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \quad (3.7)$$

where  $p = |\mathbf{p}|$  is the magnitude of the momentum and  $\mathbf{1}$  is the unit  $(D-2) \times (D-2)$  matrix.

Denoting by  $\mathbf{p}^-$  and  $\mathbf{p}^+$  the momenta before and after the collision it is straightforward that the law of specular reflection may be expressed as

$$\mathbf{p}^+ = \mathbf{p}^- - 2(\hat{\mathbf{n}} \cdot \mathbf{p}^-) \hat{\mathbf{n}} \quad (3.8)$$



**Figure 4.** The coordinate system used for describing the monodromy matrix of scattering from a sphere. The coordinate  $r_{\parallel}$  is along the orbit,  $r_{\perp}$  is perpendicular to it and is in the plane defined by the two segments of the orbit, and  $r_b$  represents the remaining coordinates which are orthogonal both to  $r_{\parallel}$  and  $r_{\perp}$ .

where  $\hat{n}$  is a unit vector which is perpendicular to the surface at the point of collision. From this law it follows that the monodromy matrix corresponding to a collision with a sphere of radius  $R$  is

$$\mathcal{M}_{\text{collision}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ \frac{-p}{2R \cos \theta} - 1 & 0 & 0 & 0 \\ 0 & 0 - \mathbf{1} & 0 & 0 \\ 0 & 0 & \frac{-p \cos \theta}{2R} \mathbf{1} - \mathbf{1} & 0 \end{pmatrix} \quad (3.9)$$

where  $\theta$  is the angle between  $\hat{n}$  and  $p^-$ . The derivation of this formula is presented in the appendix. The absolute values of eigenvalues  $\mathcal{M}_{\mu}^{(i)}$  can be calculated straightforwardly using the block form of the above matrices and the result is:

$$\Lambda_v^{+(i)} = 1 + \frac{r_i}{R \cos \theta} + \sqrt{\left(1 + \frac{r_i}{R \cos \theta}\right)^2 - 1} \simeq \frac{2r_i}{R \cos \theta} \quad v = 1 \quad (3.10)$$

$$\Lambda_v^{+(i)} = 1 + \frac{r_i \cos \theta}{R} + \sqrt{\left(1 + \frac{r_i \cos \theta}{R}\right)^2 - 1} \quad v = 2, 3, \dots (D-1)$$

where  $r_i$  is the length of the  $i$ -segment of the orbit and  $\theta$  is the angle shown in figure 4. In the two-dimensional case, there is only one eigenvalue  $\Lambda_1^+$ . Thus the product of these

eigenvalues satisfies

$$\prod_{v=1}^{D-1} \Lambda_v^{+(i)} = \eta_i \left( \frac{r_i^{D-1}}{\sigma} \right) \quad (3.11)$$

where  $\sigma = \mathcal{V}_{D-1} R^{D-1}$  is the cross section of the hypersphere with  $\mathcal{V}_k$  equal to the volume of a  $k$ -dimensional unit sphere

$$\mathcal{V}_k = \frac{\pi^{k/2}}{\Gamma(1 + k/2)} \quad (3.12)$$

and

$$\eta_i \simeq \frac{2\mathcal{V}_{D-1}}{\cos\theta} \left( \cos\theta + R/r_i + \sqrt{\cos^2\theta + 2R\cos\theta/r_i} \right)^{D-2}. \quad (3.13)$$

This result suggests that the amplitude (3.5) can be written in the form

$$|\det(\mathcal{M}_\mu - \mathbf{1})|^{-1} = \prod_{i=1}^j \xi_i \frac{\sigma}{r_i^{D-1}} \quad (3.14)$$

where  $\xi_i$  is a geometric factor which may be considered as a random variable, at least for long enough orbits. Notice that  $\xi_i$  is not exactly  $1/\eta_i$ . The reason is that the monodromy matrices  $\mathcal{M}_\mu^{(i)}$  do not commute therefore the eigenvalues of their product is not equal to the product of the eigenvalues of each matrix. Using expression (3.4) for the number of orbits and (3.14) for the amplitude, one can write (3.3) in the form:

$$\langle K_j(s) \rangle = \frac{-\beta}{2\pi^2 j l^j} \frac{\partial^2}{\partial s^2} \Re \int \cdots \int \prod_{i=1}^j (dr_i d\Omega_i \xi_i e^{-r_i/l}) \hat{\delta} \left( \sum_i \mathbf{r}_i \right) e^{is \sum_i r_i/v} \quad (3.15)$$

where we have used the relation  $l = 1/N\sigma$  which relates the classical mean free path to the density of the impurities and their cross section. In this expression, the constraint that the orbit is periodic is imposed by the periodic delta function  $\hat{\delta}(\mathbf{r})$ . It is defined on the torus as

$$\hat{\delta}(\mathbf{r}) = \sum_{\mathbf{n}'} \delta(\mathbf{r} - \mathbf{n}') \quad (3.16)$$

where  $\mathbf{n}'$  is a  $D$ -dimensional integer vector, and the sum runs over all its values. The prefactor  $1/j$  was introduced to avoid multiple counting of the same periodic orbit (in the construction explained above each periodic orbit may start at any of the  $j$  impurities which it is visiting hence the same periodic orbit is constructed  $j$  times). The time  $t_\mu$  in (3.3) is equal to  $\sum_i r_i/v$ , where  $v$  is the magnitude of the velocity of the particle.

Denoting by  $\langle \cdots \rangle_\Omega$  an average over all angles  $\Omega_i$ ,

$$\langle \cdots \rangle_\Omega = \int \cdots \int \prod_{i=1}^j \frac{d\Omega_i}{D\mathcal{V}_D} (\cdots) \quad (3.17)$$

we obtain

$$\langle K_j(s) \rangle \simeq \frac{-\beta}{2\pi^2 j} \frac{\partial^2}{\partial s^2} \left( \frac{\xi}{l} \right)^j \Re \int \cdots \int \prod_{i=1}^j (dr_i e^{-r_i/l + ir_i s/v}) \left\langle \hat{\delta} \left( \sum_i \mathbf{r}_i \right) \right\rangle_\Omega. \quad (3.18)$$

Here we assumed that the product of the geometric factors,  $\xi_i$ , has a simple scaling property,  $\xi^j = (D\mathcal{V}_D)^j \prod \xi_i$ , where  $\xi$  is the typical value of  $D\mathcal{V}_D \xi_i$  independent of the angles  $\Omega_i$ . It is plausible that for long orbits  $\xi$  is given by

$$\xi = D\mathcal{V}_D/\bar{\eta} \quad (3.19)$$

where  $\bar{\eta}$  denotes a proper angle averaging of  $\eta_i$  given by (3.13). This is justified for long orbits since the eigenvalues of the monodromy matrix of these orbits are determined by the Lyapunov exponents of the system. The Lyapunov exponents self-average, and for long orbits they take ensemble averaged values. Thus for long orbit (3.14) can be written as  $|\det(\mathcal{M}_\mu - \mathbf{1})| = \langle |\det(\mathcal{M}_\mu - \mathbf{1})| \rangle = \langle \prod_{i=1}^j \eta_i r_i^{D-1} / \sigma \rangle$ . For strong instability  $\eta_i$  can be considered as independent random variables with average  $\bar{\eta}$ . Therefore for long orbits  $\xi_i = 1/\bar{\eta}$  which leads to (3.19). The average of the periodic delta function can be calculated using the Poisson summation formula, and the integral  $\int_0^\pi e^{iz \cos \theta} \sin^{2\nu} \theta d\theta = \sqrt{\pi} \Gamma(\nu + 1/2) (2/z)^\nu J_\nu(z)$ , where  $J_\nu$  is the Bessel function. The result is

$$\left\langle \hat{\delta} \left( \sum_{i=1}^j r_i \right) \right\rangle_\Omega = \sum_{\mathbf{m}} \prod_{i=1}^j \mathcal{G}_D(2\pi |\mathbf{m}| r_i) \tag{3.20}$$

where  $\mathbf{m}$  is an integer vector and  $|\mathbf{m}|$  is its modulus, namely the square root of the sum of its  $D$  components squared. The function  $\mathcal{G}_D(z)$  is given by

$$\mathcal{G}_D(z) = \Gamma\left(\frac{D}{2}\right) \left(\frac{2}{z}\right)^{D/2-1} J_{D/2-1}(z). \tag{3.21}$$

It is normalized so that  $\mathcal{G}_D(0) = 1$  for all  $D$ . This follows, e.g., from the asymptotic expansion of the Bessel functions at small arguments. For instance, in the two-dimensional case  $\mathcal{G}_2(z) = J_0(z)$ , while for  $D = 3$  it is  $\mathcal{G}_3(z) = \sin(z)/z$ .

Our next step is to perform the sum over the index  $j$  which counts the number of impurities the orbit is bounced from. After changing variables in (3.18) to  $z = r_i/l$ ,  $\langle K_j(s) \rangle$  takes the form

$$\langle K_j(s) \rangle = \frac{-\beta}{2\pi^2} \frac{\partial^2}{\partial s^2} \Re \sum_{\mathbf{m}} \frac{1}{j} \left[ \xi \int_0^\infty dz e^{-z(1-i\tau s)} \mathcal{G}_D(2\pi |\mathbf{m}| lz) \right]^j \tag{3.22}$$

where  $\tau = l/v$  is the elastic mean free time. The sum over  $j$  in (3.2) can be performed now. Notice that the first non-vanishing term in this sum is  $j = 2$ . The reason is that on the torus geometry, where there are no boundaries, there are no closed orbits which are scattered only once from an impurity. Thus, the sum over  $j$  yields

$$\langle K_P^{(c)}(s) \rangle = \frac{\beta}{2\pi^2} \frac{\partial^2}{\partial s^2} \Re \sum_{\mathbf{m}} \left\{ \ln \left[ 1 - \xi \hat{\mathcal{G}}_D(2\pi |\mathbf{m}| l, 1 - i\tau s) \right] + \xi \hat{\mathcal{G}}_D(2\pi |\mathbf{m}| l, 1 - i\tau s) \right\} \tag{3.23}$$

where  $\hat{\mathcal{G}}_D(x, y)$  is the Laplace transform of  $\mathcal{G}_D(xz)$ , namely

$$\begin{aligned} \hat{\mathcal{G}}_D(x, y) &= \int_0^\infty dz e^{-zy} \mathcal{G}_D(xz) \\ &= \frac{1}{\sqrt{x^2 + y^2}} F\left(\frac{1}{2}, \frac{D}{2} - 1; \frac{D}{2}; \frac{x^2}{x^2 + y^2}\right). \end{aligned} \tag{3.24}$$

where  $F(\alpha, \beta; \gamma; \delta)$  is the hypergeometric function. The function  $\hat{\mathcal{G}}_D(x, y)$  is usually non-analytic and multivalued. The correct branch in the complex plane should be chosen such

that it satisfies the relation  $\hat{\mathcal{G}}_D(0, y) = 1/y$ . For some specific cases

$$\hat{\mathcal{G}}_D(x, y) = \begin{cases} \frac{y}{(x^2 + y^2)} & D = 1 \\ \frac{1}{\sqrt{x^2 + y^2}} & D = 2 \\ \frac{1}{x} \tan^{-1} \left( \frac{x}{y} \right) & D = 3 \\ \frac{2}{x^2} (\sqrt{x^2 + y^2} - y) & D = 4 \end{cases} \quad (3.25)$$

where the case  $D = 1$  should be understood as the quasi-one-dimensional case. By quasi-one-dimensional geometry we refer to a  $D$ -dimensional box which is extended in one direction and has length 1, while in all other perpendicular directions its size is  $L$ , where  $L \ll 1$ . In addition, it is assumed that the system is observed over a time which is much larger than the time that it takes for the particle to move a distance  $L$ . The scatterers are assumed to be still small, namely  $R \ll L$ .

We shall now argue that the parameter  $\xi$  equals unity. For this purpose, let us consider the limit  $s \rightarrow 0$ , and  $l \ll 1$  fixed. This limit corresponds to strong scattering and long time such that the particle already explores all its available phase space. The contribution from orbits which are not scattered from impurities practically vanishes in this limit, and the result for  $K_P(s)$  is known from random matrix theory. It is  $K_P(s) = -\beta/2\pi^2 s^2$  independent of  $D$ . The leading contribution, in this case, comes from the null vector  $\mathbf{m} = 0$ . Thus from the properties of  $\hat{\mathcal{G}}_D(x, y)$  it follows that

$$\lim_{s, \tau \rightarrow 0} \langle K_P^{(c)}(s) \rangle = \frac{\beta}{2\pi^2} \Re \frac{\partial^2}{\partial s^2} \left\{ \ln \left( \frac{(1 - \xi) - i\tau s}{1 - i\tau s} \right) + \frac{\xi}{1 - i\tau s} \right\} \quad (3.26)$$

and the correct behaviour is obtained only if  $\xi = 1$ .

The above argument holds in any dimension, yet for the two-dimensional case there is an alternative way to see that  $\xi = 1$ . It is based on the relation (3.19), where a proper averaging of  $\eta = 4/\cos \theta$ , defined in (3.13), over the scattering angle  $\theta$  is performed (see figure 4). Suppose that the impact parameter of a typical trajectory, which is scattered from a disc, is a random variable distributed uniformly between  $-R$  and  $R$ . Then it is straightforward to show that the probability distribution of the scattering angle  $\theta$  is  $P(\theta) = \cos(\theta)/2$ , where  $-\pi/2 \geq \theta \geq \pi/2$ . Thus

$$\bar{\eta} = \int_{-\pi/2}^{\pi/2} d\theta P(\theta) \eta(\theta) = 2\pi \quad (3.27)$$

and from (3.19) it follows immediately that  $\xi$  equals unity.

#### 4. The spectral determinant $\mathcal{D}(s)$

As was discussed in the introduction, the essential quantity in the calculation of the spectral density correlator and its Fourier transform, the form factor, is the classical spectral determinant  $\mathcal{D}(s)$ . This function satisfies the relation [33]

$$\mathcal{D}(s) = Z(is)Z(-is) \quad (4.1)$$

where  $1/Z(is)$  is the classical dynamical zeta function [38]. The dynamical zeta function is the spectral determinant, associated with the Perron–Frobenius operator, which characterizes the classical dynamics of a chaotic system. Its zeros,  $z_n$ , are the eigenvalues corresponding to

the modes in which a disturbance in the particles density decays into its ergodic distribution. For Anosov systems this decay is exponential in time, namely  $e^{-z_n t}$ . Closed Hamiltonian systems always contain the eigenvalue  $z_0 = 0$  which corresponds to conservation of probability. The probability density mode associated with this eigenvalue is the ergodic distribution. The normalization of  $1/Z(z)$  is chosen to satisfy  $\lim_{z \rightarrow 0} zZ(z) = 1$ .

The relation (1.8) enables us to identify the dynamical zeta function. Since  $K_P(s)$  is a sum of the ‘integrable’ part (2.27) and the chaotic part (3.23) it follows that

$$Z(z) = Z_i(z)Z_c(z) \tag{4.2}$$

where  $Z_i(z)$  is associated with the ‘integrable’ dynamics and is given by

$$1/Z_i(z) = \left(\frac{2}{e}\right)^{4\pi/\tau} \exp\left\{-\frac{2\pi}{\tau}(2-z\tau)[\ln(2-z\tau)-1]\right\}. \tag{4.3}$$

It is normalized such that  $Z_i(0) = 1$ . The second term  $Z_c(z)$ , coming from orbits which are scattered by impurities, is

$$1/Z_c(z) = \prod_m B_m \left[1 - \hat{G}_D(2\pi|m|l, 1 - \tau z)\right] \exp\left[\hat{G}_D(2\pi|m|l, 1 - \tau z)\right]. \tag{4.4}$$

Here  $B_m$  are regularization factors which are introduced to make the product converge, and to satisfy the normalization property  $\lim_{z \rightarrow 0} zZ(z) = 1$ .

In the quasi-one-dimensional case, the dynamical zeta function can be expressed in terms of simple trigonometric functions. Using the relations  $\sin x = x \prod_{m=1}^{\infty} (1 - x^2/\pi^2 m^2)$  and  $\coth x = \sum_{m=-\infty}^{\infty} x/(x^2 + \pi^2 m^2)$ , one obtains

$$1/Z_c(z) = \mathcal{N} \exp\left\{\frac{1}{2l} \coth\left(\frac{1-z\tau}{2l}\right)\right\} \left[\frac{\sin\left(\frac{\sqrt{z\tau-z^2\tau^2}}{2l}\right)}{\sinh\left(\frac{1-z\tau}{2l}\right)}\right]^2 \tag{4.5}$$

where

$$\mathcal{N} = \frac{1}{\tau} \exp\left\{-\frac{1}{2l} \coth\left(\frac{1}{2l}\right)\right\} \left[2l \sinh\left(\frac{1}{2l}\right)\right]^2. \tag{4.6}$$

The analytic structure of this function is depicted in figure 5(a). It has essential singularities at  $z_n = (1 + i2\pi ln)/\tau$ , double poles at  $z_j = (1 + i2\pi lj)/\tau$ , double zeros at  $z_m^{(\pm)} = [1/2 \pm \sqrt{1/4 - (2\pi lm)^2}]/\tau$ , simple zero at the origin, and simple pole at  $z = 1/\tau$ . Here  $j$ ,  $m$ , and  $n$  are integers, where  $j \neq 0$  and  $m > 0$ .

When  $l > 1/4\pi$  all the zeros, except the one at the origin, are located along the line  $\Re z = 1/2\tau$ . In the other limit,  $l \ll 1/4\pi$ , there are many real zeros near the origin  $z_m^{(-)} \approx (2\pi lm)^2/\tau$ . These zeros are associated with the diffusion modes of the system. In order to show that, consider the perturbative part of the form factor in the quasi-one-dimensional case. Calculating it from the Fourier integral of expression (1.8) one can show that it is approximately given by

$$\langle K_P(t) \rangle \simeq \frac{\beta}{2\pi} |t| \sum_{m=-\infty}^{\infty} e^{-z_m^{(-)}|t|} \tag{4.7}$$

where the contributions from the poles and the essential singularities are neglected, since they are located far from the real  $s$  axis when  $\tau \rightarrow 0$ . Substituting  $z_m \approx (2\pi lm)^2/\tau$  and approximating the sum over  $m$  by an integral from  $-\infty$  to  $+\infty$  yields

$$\langle K_P(t) \rangle \simeq \frac{\beta}{2\pi} \sqrt{\frac{\tau|t|}{4\pi l^2}}. \tag{4.8}$$



This is the Altshuler–Shklovskii [7] result of the form factor of quasi-one-dimensional diffusive systems written here in terms of dimensionless quantities. The influence of the complex zeros and poles is manifested only on time scales shorter than or of order  $\tau$  which here is assumed to be very small.

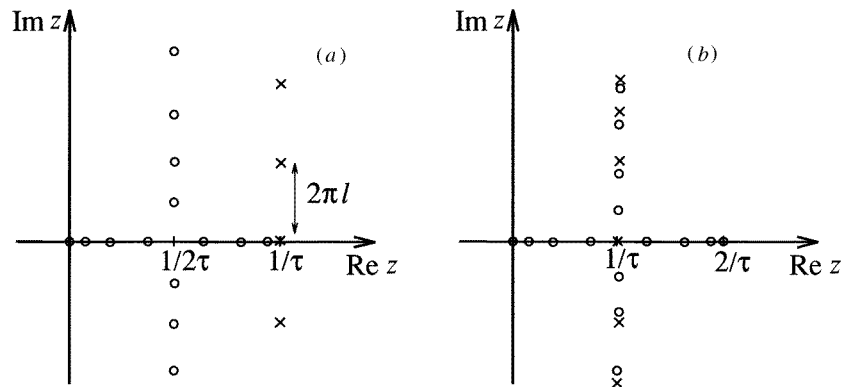
In the two-dimensional case the regularization factors  $B_m$  may be chosen to be

$$B_m^{-1} = \left[ 1 - \hat{G}_2(2\pi l|\mathbf{m}|, 1) \right] \exp \left[ \hat{G}_2(2\pi|\mathbf{m}|l, 1) \right] \tag{4.9}$$

where  $\mathbf{m} \neq \mathbf{0}$ , while  $B_0 = 1$ . It is easy to verify that with this choice, the product (4.4) converges. Therefore the zeros of  $1/Z(z)$  are those of the various terms in the product. The zeros are thus located at  $z_m^{(\pm)} = \left[ 1 \pm \sqrt{1 - (2\pi l|\mathbf{m}|)^2} \right] / \tau$ . When  $l > 1/2\pi$  all of them, except the one at the origin, are complex and located along the line  $\Re z = 1/\tau$ . In the other limit,  $l \ll 1/2\pi$ , there are many real zeros associated with diffusive modes of the system,  $z_m^{(-)} \approx (2\pi l\mathbf{m})^2/2\tau$ . A straightforward calculation, similar to the one which led to (4.8), shows that these zeros lead to the Altshuler–Shklovskii result of the form factor in two dimensions,  $\langle K_P(t) \rangle \simeq \beta\tau/4\pi^2 l^2$ . The poles are located at  $z_j^{\pm} = [1 \pm i2\pi l|j|]/\tau$ , and there are branch cuts connecting  $z_j^{(+)}$  and  $z_j^{(-)}$ . The essential singularities are at  $[1 \pm i2\pi l|\mathbf{n}|]/\tau$ . In contrast to the quasi one-dimensional case, they are not equidistant along the line  $\Re z = 1/\tau$ . The analytic structure described here is shown in figure 5(b).

In higher dimensions,  $D > 2$ , the situation is not completely clear. It is not obvious that the form of regularization, used for  $D = 2$ , does apply, therefore it is difficult to extract the zeros and poles. Ignoring convergence problems, one can show that in the diffusive regime,  $l \ll 1$  there are real zeros,  $z_m^{(-)} \approx (2\pi l|\mathbf{m}|)^2/D\tau$ , in the vicinity of the origin, and that for  $D = 3$  and  $D = 4$  there are no complex zeros of  $1 - \hat{G}_D(2\pi|\mathbf{m}|l, 1 - \tau z)$  for any value of  $l$ .

Formula (1.7) for the spectral density correlator is applicable in the limit where the real part,  $z_1'$ , of the first non-vanishing zero of  $1/Z(z)$  is sufficiently large compared to unity. Near the diffusive regime, where  $l \ll 1$ , the smallest non-vanishing zero is  $z_1 = (2\pi l)^2/D\tau$ . Identifying  $l^2/D\tau$  with the diffusion constant of the particle in the system, it follows that  $z_1 = 4\pi^2/\tau_c$ , where  $\tau_c$  is the Thouless time measured in units of  $\hbar/\Delta$ . Therefore, up to a constant of  $4\pi^2$ ,  $z_1$  is the dimensionless conductance of the system  $g$ . When  $g \rightarrow \infty$ ,  $\mathcal{D}(s)$  reduces to  $1/s^2$  and the RMT result for the spectral density correlator, namely



**Figure 5.** The analytic structure of  $1/Z_c(z)$  for (a) quasi-one-dimensional case (b) two-dimensional case.  $\circ$  and  $\times$  represent zeros and singularities respectively. In the two-dimensional case there are also branch cuts with branch points at the singularities.

$\langle K(s) \rangle = -\sin^2(\pi s)/\pi^2 s^2$  ( $s \neq 0$ ), is recovered. The reason is that in this limit all the zeros and poles of  $\mathcal{D}(s)$ , except the zero at the origin, are far from the real  $s$  axis, therefore their contribution becomes small. Formula (1.7) with the above form of the spectral determinant  $\mathcal{D}(s)$  enables us to calculate the deviations from the RMT universal result when  $g$  is finite but sufficiently large.

### 5. The two-dimensional case, $D = 2$

In this section we shall study in more detail the two dimensional case ( $D = 2$ ), and in particular systems which belong to the unitary ensemble. Here we prefer to present most of the results in terms the form factor and to compare them with the RMT universal function which in the unitary case takes the form [4]:

$$K(t) = \begin{cases} \frac{|t|}{2\pi} & |t| \leq 2\pi \\ 1 & \text{otherwise.} \end{cases} \tag{5.1}$$

The nature of the classical dynamics of the system is determined by the elastic mean free path  $l$ . When  $l$  is larger than unity the system is in the ballistic regime, while when  $l \ll 1/2\pi$  the dynamics is diffusive. In terms of the dynamical zeta function, these two cases correspond, respectively, to the situations where  $1/Z(z)$  has only complex zeros besides the one at the origin, or also many real zeros near the origin. As will be shown this is reflected in the form of the non-universal features which decorate the universal result (5.1). In what follows we shall study the full range of the parameter  $l$ .

Let us start with the ballistic regime  $l \gg 1$ . It is convenient here to split the contribution of the term  $\mathbf{m} = \mathbf{0}$  in (3.23) from the others, and to write the perturbative part of the density correlator, associated with orbits which are scattered from impurities, as

$$\langle K_p^{(c)}(s) \rangle = \frac{\beta}{4\pi^2} \frac{\partial^2}{\partial s^2} [\mathcal{F}_0 + \mathcal{F} + \text{cc}] \tag{5.2}$$

where

$$\mathcal{F}_0 = \ln \left( 1 - \frac{1}{1 - is\tau} \right) + \frac{1}{1 - is\tau} \tag{5.3}$$

while  $\mathcal{F}$  is the contribution from  $\mathbf{m} = (m_1, m_2) \neq (0, 0)$ . An estimate for  $\mathcal{F}$ , that is crude in the ballistic regime, can be obtained replacing the sum over  $\mathbf{m}$  by an integral,

$$\mathcal{F} \approx 4 \int_1^\infty dm_1 \int_1^\infty dm_2 \ln[1 - \hat{\mathcal{G}}_2(2\pi l|\mathbf{m}|, 1 - is\tau)] + \hat{\mathcal{G}}_2(2\pi l|\mathbf{m}|, 1 - is\tau) \tag{5.4}$$

where  $\hat{\mathcal{G}}_2(x, y)$  is given by (3.25). After some algebraic manipulations it takes the form

$$\mathcal{F} \approx \frac{1}{4\pi l^2} \lim_{\Lambda \rightarrow \infty} \int_{(2\pi)^2}^\Lambda du \left\{ \ln \left[ 1 - \frac{1}{\sqrt{u + (1 - is\tau)^2}} \right] + \frac{1}{\sqrt{u + (1 - is\tau)^2}} \right\}. \tag{5.5}$$

The required indefinite integral is elementary:

$$\begin{aligned} & \int du \left\{ \ln \left[ 1 - \frac{1}{\sqrt{u + y^2}} \right] + \frac{1}{\sqrt{u + y^2}} \right\} \\ &= \left[ (u + y^2 - 1) \ln(\sqrt{u + y^2} - 1) - (u + y^2) \ln(\sqrt{u + y^2}) + \sqrt{u + y^2} \right]. \end{aligned} \tag{5.6}$$

The contribution of the upper limit of integration  $\Lambda$ , although diverging in the limit  $\Lambda \rightarrow \infty$ , is not important since its second derivative with respect to  $s$  vanishes and therefore it does

not contribute to  $\langle K_P^{(c)}(s) \rangle$ . For this reason it will be omitted. The contribution from the lower limit gives

$$\mathcal{F}(x) \approx \frac{1}{4\pi l^2} [(x^2 - 1) \ln(x - 1) - x^2 \ln x + x] \quad (5.7)$$

where

$$x = \sqrt{(2\pi l)^2 + (1 - is\tau)^2}. \quad (5.8)$$

The final result for the perturbative part of the density correlator, which includes both contributions (2.27) and (3.23), is thus

$$\langle K_P(s) \rangle = \frac{\beta}{2\pi^2} \left\{ \frac{4\pi\tau}{4 + (s\tau)^2} - \frac{1}{s^2} - \tau^2 \frac{1 - (s\tau)^2}{(1 + (s\tau)^2)^2} - 2\tau^2 \frac{1 - 3(s\tau)^2}{(1 + (s\tau)^2)^3} + \frac{1}{2} [\mathcal{F}'' + \text{CC}] \right\} \quad (5.9)$$

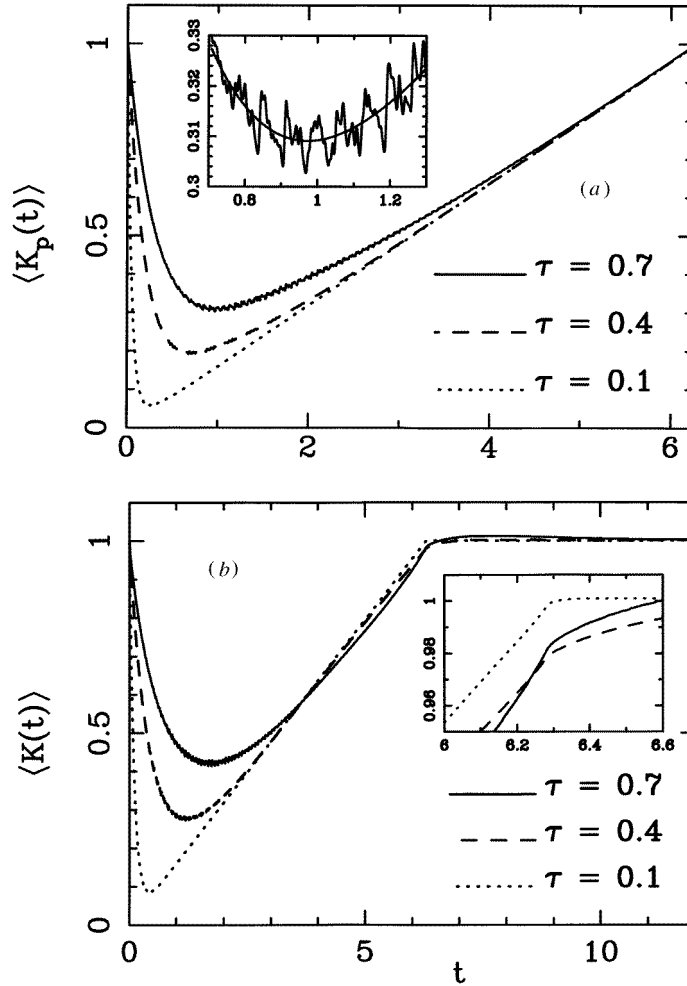
where  $\mathcal{F}''$  denotes the second derivative of  $\mathcal{F}$ , given by (5.7), with respect to  $s$ . In most of the ballistic regime where  $l \gg 1$  this term is negligible since it is divided by a large number  $4\pi l^2$ . The first term in the above expression is the contribution from the orbits which are not scattered by impurities (2.27). The Fourier transform of this term,  $\langle K_P^{(i)}(t) \rangle$  given by (2.26), is valid only for times  $t$  which are sufficiently large compared to the time of flight across the sample  $\tau_f$ . For  $t$  of order of  $\tau_f$  the perturbative part of the form factor is composed of  $\delta$ -peaks located at times which are the periods of the orbits. Only when  $t$  is sufficiently large can these  $\delta$ -peaks be smeared out to give (2.26). Beside this term, formula (5.9) is identical to that obtained by Altland and Gefen [23] (equation (32) there in the limit of vanishing smearing of energy levels). In various regimes of the parameter  $s$  it reduces to,

$$\langle K_P(s) \rangle \sim \frac{\beta}{2\pi^2} \begin{cases} -\frac{1}{s^2} & \tau \ll \frac{1}{s} \\ \frac{4\pi}{s^2\tau} + \frac{3}{\tau^2 s^4} & \tau_f \ll \frac{1}{s} \ll \tau \quad l \gg 1. \\ \frac{-1}{4\pi l^2 s^2} & \frac{1}{s} \ll \tau_f \end{cases} \quad (5.10)$$

The above three domains of the parameter  $s$  are referred to as B1, B2, and B3 in [23]. The first of them corresponds to long time scales over which the motion of the particle is already ergodic. The result is therefore the universal one. In the second regime, the time is smaller than elastic mean free time but longer than the time of flight. Here, depending on the actual value of the parameter  $\tau$ , the contribution from orbits which do not scatter from impurities may be the dominant one. The last  $s$  domain corresponds to time scales that are smaller than the time required to cross the sample. Here the contribution to the density correlator comes only from short orbits which are scattered by impurities, and the result is only a crude estimate.

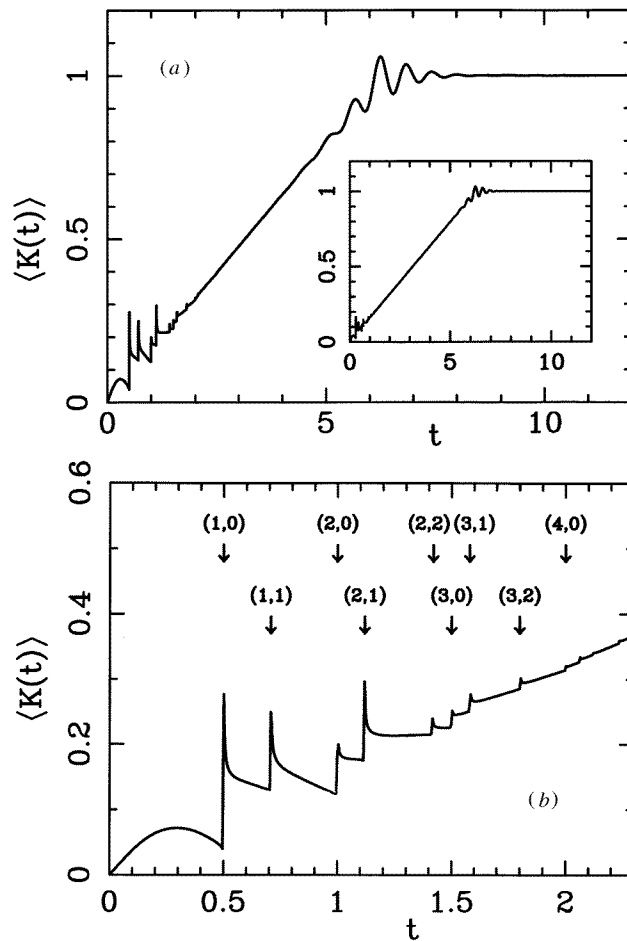
We shall now present a numerical study of the form factor of a system belonging to the unitary ensemble. Here the form factor is calculated from the Fourier transform (1.2) of formula (1.7) for the spectral density correlator. The spectral determinant  $\mathcal{D}(s)$  given by (4.1), (4.3) and the product (4.4) is numerically calculated by including the terms with  $|m_i| \leq 1000$  ( $i = 1, 2$ ), and using the regularization (4.9).

In figure 6 we present the results for the form factor in the ballistic regime where the elastic mean free path  $l = 10$  is larger by an order of magnitude than the size of the system. Here, the contribution from the orbits which are not scattered by impurities is large, while the effect of  $\mathcal{F}$  is negligible.



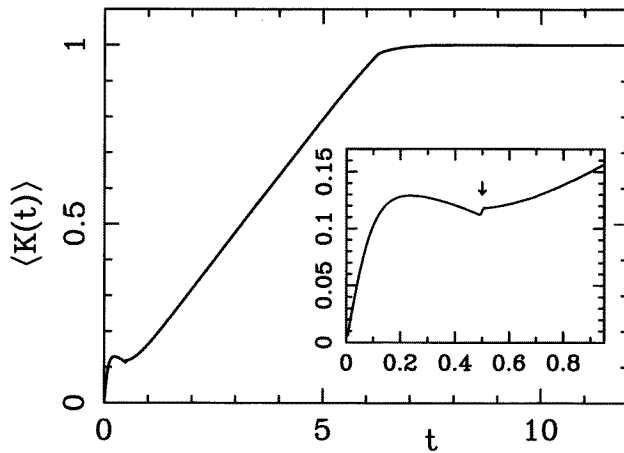
**Figure 6.** (a) The perturbative part of the form factor for  $l = 10$ . The inset is a magnification of the region  $0.7 \leq t \leq 1.3$  for the case  $\tau = 0.7$  and  $l = 10$ . (b) The full form factor for the same parameters as in (a). The inset is a magnification of the vicinity of the Heisenberg time.

In figure 6(a) we present the perturbative part of the form factor for various values of  $\tau$ . A magnification of the domain  $0.7 \leq t \leq 1.3$  is the insert for the case  $\tau = 0.7$  where the time of flight is  $\tau_f = 0.07$ . The smooth curve corresponds to a calculation in which the  $\mathcal{F}$  term in (5.9) has been omitted. Several conclusions follow from this figure: (i) the contribution from orbits which are not scattered by impurities can be important, depending on the values of  $\tau$ , (ii) the effect of  $\mathcal{F}$ , i.e. the contribution from non-vanishing  $m$  terms, is small in the ballistic regime, and (iii) there is a minimum of the form factor at  $t$  of order  $\tau$ . In figure 6(b) the full form factor is depicted. It was calculated under the assumption that formula (1.11) for the non-perturbative part of the density–density correlator can be still used with the approximation (4.3) for the part of the spectral determinant associated with orbits which are not scattered by impurities. The results of figure 6 are therefore valid only when  $t \gg \tau_f$  and  $|t - 2\pi| \gg \tau_f$ . This figure shows that the singularity of the RMT universal form factor (5.1), at the Heisenberg time, is smoothed out.

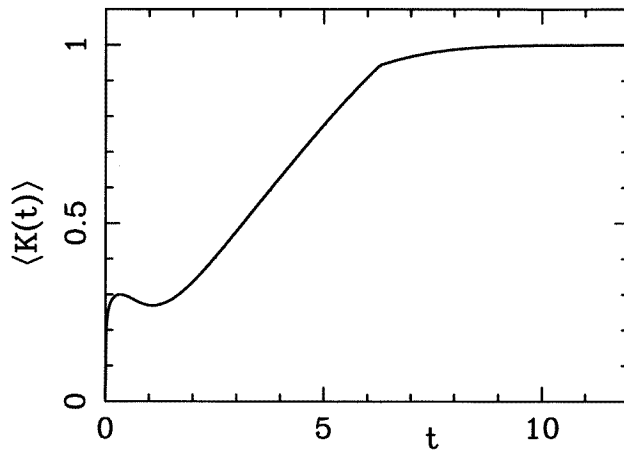


**Figure 7.** (a) The form factor for the parameters  $l = 0.5$  and  $\tau = 0.25$ . The inset is the form factor for  $l = 0.5$  and  $\tau = 0.15$ . (b) A magnification of the domain  $t \leq 2.3$ . The times indicated by arrows are the periods of the orbits which are not scattered by impurities. The pair  $(M_x, M_y)$  above each arrow represents the winding numbers, namely the topology.

The form factor in an intermediate regime where  $l = 0.5$  is depicted in figure 7. Here  $\tau = 0.25$ , and hence  $\tau_f = 0.5$ . Inserted is the result for  $l = 0.5$ ,  $\tau = 0.15$ . The contribution of  $\langle K^{(i)}(t) \rangle$  has been ignored since it is meaningless when  $\tau < \tau_f$ . The deviations from universality present themselves in two time domains. Near the origin where  $t \approx \tau_f$ , and in the vicinity of the Heisenberg time  $t \approx 2\pi$ . Near the origin  $\langle K(t) \rangle$  shows a singular behaviour. The spikes which appear in  $\langle K(t) \rangle$  result from the underlying torus geometry. They can be associated with short orbits that are scattered from a very small number of impurities, and therefore still preserve the topology of the orbits of the clean system. The positions of the spikes are at the periods of the periodic orbits which are not scattered by impurities. Their amplitudes decay as  $e^{-t/\tau}$  since the probability for the existence of such orbits decays exponentially with time. In figure 7(b) a magnification of figure 7(a) for the domain  $0 \leq t \leq 2.3$ , ( $\tau = 0.25$ ), is presented. The times indicated by arrows and the winding numbers pairs  $(M_x, M_y)$  are the periods of the orbits which are not scattered by impurities.



**Figure 8.** The form factor for the parameters  $l = 5^{1/2}/6\pi$  and  $\tau = l/2$ . The inset is for the time domain  $0 < t < 0.9$ .



**Figure 9.** The form factor in the diffusive regime for the parameters  $l = \frac{1}{40}$ ,  $\tau = \frac{1}{200}$  leading to  $\tau_f = \frac{1}{5}$ .

Near the Heisenberg time, the universal RMT behaviour (5.1) is completely changed. The singularity disappears and is replaced by oscillations with a period approximately equal to the time of flight  $\tau_f$ . The parameters in figure 7 were chosen such that  $1/Z(z)$  has no real zeros except the one at the origin. This fact is reflected in the oscillatory behaviour of the form factor near the Heisenberg time. The real part of the first non-vanishing zero is  $z'_1 = 1/\tau = 4$ . Thus the universal RMT limit (5.1) of  $\langle K(t) \rangle$  is expected to be reached when  $\tau$  decreases. This is demonstrated in the inserted figure where  $z'_1 = 1/\tau = \frac{20}{3}$ . It shows that as  $\tau$  decreases, the time intervals over which the non-universal features are large shrink, and their amplitude decreases.

The oscillatory behaviour of the form factor near the Heisenberg time changes when  $1/Z(z)$  has real zeros in addition to the one at origin and the complex ones. In figure 8 the form factor calculated for the parameters  $l = 2\tau = \sqrt{5}/6\pi \approx 0.12$ ,  $\tau_f = \frac{1}{2}$  is presented. For these parameters the first non-vanishing zero is  $z_1 = 4/3l \simeq 11.23$ , the second one is at  $8/3l$ ,

and all the others are complex. Comparing figures 7 and 8, one sees that the singularities, which appear in figure 7 near the origin, almost disappear, and the oscillatory behaviour near the Heisenberg time becomes smooth. Inserted in this figure is a magnification of the time domain  $0 < t < 0.9$ . The jump indicated by the arrow is reminiscent of the toroidal topology of the system. It appears at the period of the shortest orbit of the clean system  $[(M_x, M_y) = (1, 0) \text{ and } (0, 1)]$ .

The structure of the non-universal features does not change much as  $l$  further decreases. In figure 9 we present the form factor for  $l = \frac{1}{40}$  that is much deeper in the diffusive regime.  $\tau$  is set to be equal to  $\frac{1}{200}$  so that the time of flight is  $\tau_f = \frac{1}{5}$ . For these parameters, the smallest real zero is  $z_1 \approx 2.483$  and there are approximately 127 real zeros in the interval  $[0, 1/\tau)$ . A similar number of zeros exist in the interval  $(1/\tau, 2/\tau]$ , and all the other ones are complex.

## 6. Summary and discussion

In this paper the spectral density correlator and the form factor of a system in which the amount of disorder is controlled was analysed with the help of periodic orbit theory. Two parameters govern the behaviour of these functions:  $l$  the elastic mean free path, and  $\tau$  the elastic mean free time. The elastic mean free path measures the amount of disorder in the system. When  $l > 1$ , i.e. when the elastic mean free path is larger than the system size, the dynamics is ballistic, while it is diffusive when  $l \ll 1$  [23, 39]. The elastic mean free time determines the classical time scales of the system, since  $v = l/\tau$  where  $v$  is the dimensionless velocity. As  $\tau \rightarrow 0$  other time scales effectively increase relative to  $\tau$  and the RMT universal behaviour is recovered (this will be discussed in detail in what follows).

The non-universal features, that appear for finite small values of  $\tau$ , decorate the RMT result of the form factor (5.1) mainly in two regions: near the origin  $t = 0$ , and in the vicinity of the Heisenberg time  $t = 2\pi$ . They are appreciable over small intervals that scale with  $\tau$ . Thus as  $\tau \rightarrow 0$  these time domains shrink and the universal result is reached. The nature of the non-universal behaviour of the system is determined by the elastic mean free path  $l$ . Detailed analysis of this behaviour, that was carried out for a system with two degrees of freedom belonging to the unitary ensemble, will now be summarized.

The typical structure of the form factor in the ballistic regime,  $l \gg 1$  is depicted in figure 6. For  $t < \tau$  it is dominated by the contribution,  $e^{-2t/\tau}$ , of periodic orbits that are not scattered by impurities (2.26). Near the Heisenberg time  $|t - 2\pi| < \tau$ , the RMT singularity is smoothed out. The typical line shape of the form factor in this regime is characterized by a minimum in the vicinity of  $t = \tau$ .

The form factor, in the intermediate regime  $l = 0.5$ , is depicted in figure 7. Here the contribution from periodic orbits that are not scattered by impurities is negligible, and the behaviour is determined by orbits that do scatter. Near the origin (see figure 7(b)) the form factor exhibits a singular behaviour. These singularities can be associated with short orbits that are scattered from a very small number of impurities, and therefore still preserve the topology of the orbits of the clean system. The amplitudes of the singularities decay exponentially as  $e^{-t/\tau}$  since the probability for the existence of such orbits decreases exponentially with time. The behaviour near the Heisenberg time is oscillatory with a period of order of the time of flight across the system  $\tau_f$ .

Moving towards the diffusive regime where  $l < 1/2\pi$ , the non-universal features change their character as presented in figure 8, for  $l = \sqrt{5}/6\pi \simeq 0.12$ . The singular behaviour near the origin almost disappears, and the oscillations near the Heisenberg time transform

into an overall smooth curve. The situation does not change much in the diffusive regime, as can be seen from figure 9 where  $l = \frac{1}{40}$ .

An understanding of the scenario described above was obtained by analysing the classical dynamical zeta function of the system,  $1/Z(z)$ . The zeros of this function are the eigenvalues of the Perron–Frobenius operator of the system [38]. They are thus associated with the decaying modes of the probability distribution of particles towards the ergodic distribution. The dynamical zeta function, in an arbitrary dimension, has been calculated in our model, and was expressed in terms of an infinite product (4.4). For the quasi-one-dimensional case a simple formula has been obtained in terms of simple trigonometric functions (4.5). Its structure in the complex  $z$  plane is shown in figure 5(a).

A crucial role is played by the singularities and the zeros of the dynamical zeta function, since, up to constants, the perturbative part of the form factor,  $\langle \hat{K}_P(t) \rangle$ , and non-perturbative part,  $\langle \hat{K}_{osc}(t) \rangle$ , are the Fourier transforms (in  $s = -iz$ ) of  $\frac{\partial^2}{\partial z^2} [\ln Z(z) + \ln Z(-z)]$  and  $Z(z)Z(-z) \cos(-i2\pi z)$  respectively. Therefore, the zeros and singularities that are closest to the imaginary  $z$  axis dominate the behaviour of the form factor.

The analysis of  $1/Z(z)$  in the two-dimensional case shows that its zeros associated with orbits which are scattered from impurities appear in the interval  $[0, 2/\tau]$  with  $\Im z = 0$ , and on the line  $\Re z = 1/\tau$  (see figure 5(b)). When  $l > 1/2\pi$  all of them except two zeros, one at the origin and the second at  $z = 2/\tau$ , lie on the line  $\Re z = 1/\tau$ . The singularities are located only along this line,  $\Re z = 1/\tau$ . The part of  $1/Z(z)$  associated with orbits that are not scattered has a cut and a branch point at  $z = 2/\tau$ . Since, with the exception of the zero at the origin, all these zeros and singularities scale as  $1/\tau$ , their contribution to  $\langle \hat{K}(t) \rangle$  is expected to fall off exponentially with an exponent that scales as  $1/\tau$ . It therefore vanishes rapidly in the limit  $\tau \rightarrow 0$ , and only the RMT term associated with the zero at the origin survives.

In the ballistic regime  $l \gg 1$ , the contribution from all zeros and singularities that lie at  $\Re z = 1/\tau$  is negligible. That is easier to see from (5.9) and (5.10). Thus for  $|t| < \tau$ ,  $\langle \hat{K}(t) \rangle$  is dominated by the contribution (2.26) of periodic orbits that are not scattered by impurities.

As  $l$  becomes of order unity, the contribution of non-scattered orbits becomes negligible and the behaviour is governed by orbits which do scatter. If also  $l > 1/2\pi$ , then all the zeros  $z_n$  of  $1/Z(z)$ , except those at the origin and at  $2/\tau$ , are complex and located along the line  $\Re z = 1/\tau$ . The non-universal features are therefore oscillatory, because the contribution of a zero,  $z_n$ , to  $\langle \hat{K}_P(t) \rangle$  and  $\langle \hat{K}_{osc}(t) \rangle$  is proportional to  $\exp(-z_n t)$  and  $\exp(-z_n |t - 2\pi|)$  respectively.

Zeros of  $1/Z(z)$  start to appear on the real  $z$  axis, when  $l$  becomes smaller than  $1/2\pi$ . Some of them dominate the behaviour of the form factor since they are closer to the imaginary  $z$  axis than the complex zeros. In particular the non-universal features of the form factor become non-oscillatory. This change of behaviour is demonstrated in figures 7–9.

The diffusive limit is reached when  $l \ll 1/2\pi$ . The real zeros in this limit concentrate near the origin. By summing over their contributions one recovers the Altshuler and Shklovskii result for the perturbative part of the form factor [7] which applies for times smaller than  $\tau_c/4\pi^2$  where  $\tau_c = D\tau/l^2$  is the Thouless time.

In this work the semiclassical approximation was used. It requires the wavelength to be much smaller than the radius of the spheres  $R$ , that is much smaller than the size of the system. This leads to a constraint on the time of flight,  $\tau_f \ll 1$ . Nevertheless, the contribution of the scattered trajectories to (5.9) is identical to the one found by AG [23] for  $\delta$  scatterers. The contribution of trajectories that are not scattered by impurities is, however, different for these models.



Throughout this paper the classical elastic mean free path  $l$  was used. In the semiclassical limit, for hard spheres, the quantum elastic mean free path is  $l/2$ . This is a well known result in three dimensions [40]. It can be easily extended to two dimensions [41].

Several important questions require further study. The most important one concerns the renormalization of the Perron–Frobenius eigenvalues due to quantum interference effects. In the diffusive regime this is manifested by renormalization of the diffusion constant [39] which leads to weak localization effects. These corrections, therefore, are of higher order in  $1/g$  where  $g$  is the dimensionless conductance. A deeper understanding of the analytical structure of the dynamical zeta function is desirable for a higher number of degrees of freedom  $D > 2$ . This requires a regularization of the product (4.4) that is different from the one used for the two-dimensional case (4.9). Another problem concerns the generalization of this formalism to the situation of small angle scattering, which is the common scattering event an electron in a quantum dot experiences. Finally, effects of diffraction [26], tunnelling and ghost orbits [27] were not taken into account in this work and their exploration is left for further studies.

### Acknowledgments

It is our great pleasure to thank B L Altshuler, A V Andreev, B Simons, A Altland and Y Gefen for informative and stimulating discussions on this problem. This work was supported in part by the US–Israel Binational Science Foundation (BSF), and by the Fund for Promotion of Research at the Technion.

### Appendix. The monodromy matrix $\mathcal{M}_{\text{collision}}$

This is the derivation of formula (3.9) for the monodromy matrix corresponding to elastic collision with a sphere of radius  $R$ , as illustrated in figure 4. In what follows, the superscripts  $-$  and  $+$  are used to denote quantities before and after the collision. An expansion of  $\mathbf{p}^+$  to linear order in the deviations  $\delta\mathbf{p}^-$  and  $\delta\mathbf{r}^-$ , using formula (3.8) for specular reflection, yields

$$\delta\mathbf{p}^+ = \delta\mathbf{p}^- - 2(\hat{\mathbf{n}} \cdot \delta\mathbf{p}^-)\hat{\mathbf{n}} - 2(\delta\hat{\mathbf{n}} \cdot \mathbf{p}^-)\hat{\mathbf{n}} - 2(\hat{\mathbf{n}} \cdot \mathbf{p}^-)\delta\hat{\mathbf{n}} \quad (\text{A.1})$$

where  $\delta\hat{\mathbf{n}}$  is the change in the unit vector perpendicular to the surface due to the change in the point of collision.

$$\delta\hat{\mathbf{n}} = \hat{\mathbf{n}}(\mathbf{r}^- + \delta\mathbf{r}^-) - \hat{\mathbf{n}}(\mathbf{r}^-). \quad (\text{A.2})$$

In terms of the coordinate system attached to the orbit before the collision  $(r_{\parallel}^-, r_{\perp}^-, r_b^-)$ , (see figure 4), it is easy to see that

$$\hat{\mathbf{n}} = \begin{pmatrix} -\cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \quad \text{and} \quad \delta\hat{\mathbf{n}} = \frac{1}{R} \begin{pmatrix} \delta r_{\perp}^- \tan\theta \\ \delta r_{\perp}^- \\ \delta r_b^- \end{pmatrix}. \quad (\text{A.3})$$

If we choose  $\delta\mathbf{p}^-$  such that the magnitude of  $\mathbf{p}^-$  is not changed, namely  $\delta\mathbf{p}^-$  is orthogonal to  $\mathbf{p}^-$ , we get from (A.1) and (A.3) that

$$\begin{aligned}\delta\tilde{\mathbf{p}}^+ &= \begin{pmatrix} 0 \\ \delta p_{\perp}^- \\ \delta p_b^- \end{pmatrix} - \delta p_{\perp}^- \begin{pmatrix} -\sin 2\theta \\ 2\sin^2 \theta \\ 0 \end{pmatrix} - 2p \frac{\delta r_{\perp}^-}{R} \begin{pmatrix} -\sin \theta \\ \frac{\sin^2 \theta}{\cos \theta} \\ 0 \end{pmatrix} + 2\frac{p}{R} \begin{pmatrix} \delta r_{\perp}^- \sin \theta \\ \delta r_{\perp}^- \cos \theta \\ \delta r_b^- \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \delta p_{\perp}^- \sin 2\theta \\ \delta p_{\perp}^- \cos 2\theta \\ \delta p_b^- \end{pmatrix} + 2\frac{p}{R} \begin{pmatrix} 2\delta r_{\perp}^- \sin \theta \\ \delta r_{\perp}^- \frac{\cos 2\theta}{\cos \theta} \\ \delta r_b^- \cos \theta \end{pmatrix}\end{aligned}\quad (\text{A.4})$$

where we use the symbol  $\delta\tilde{\mathbf{p}}^+$  (and not  $\delta\mathbf{p}^+$ ) to indicate that this vector is expressed in the local coordinate system before the collision. The next step is to express the result in terms of the local coordinates attached to the orbit but after the collision. In particular, in the new coordinate system

$$\delta\mathbf{r}^+ \Rightarrow -\delta\mathbf{r}^+. \quad (\text{A.5})$$

The result for  $\delta\mathbf{p}^+$ , transformed to the new local coordinate system, can be obtained by rotating the coordinate system by  $2\theta$  around an axis parallel to  $\mathbf{r}_b^-$  clockwise (see figure 4) and an inversion with respect to the origin. This can be done by multiplying the result by the matrix

$$\mathcal{R} = \begin{pmatrix} -\cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta - \cos 2\theta & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (\text{A.6})$$

Thus  $\delta\mathbf{p}^+ = \mathcal{R}\delta\tilde{\mathbf{p}}^+$  leading to

$$\delta\mathbf{p}^+ = - \begin{pmatrix} 0 \\ \delta p_{\perp}^- \\ \delta p_b^- \end{pmatrix} - \frac{2p}{R} \begin{pmatrix} 0 \\ \delta r_{\perp}^- / \cos \theta \\ \delta r_b^- \cos \theta \end{pmatrix}. \quad (\text{A.7})$$

This result and (A.5) leads to the form (3.9) of the monodromy matrix corresponding to a collision.

## References

- [1] Haake F 1991 *Quantum Chaos* (New York: Springer)
- [2] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
- [3] Giannoni M J, Voros A and Zinn-Justin J (eds) 1991 *Chaos and Quantum Physics Proc. Les-Houches Summer School, Session LII 1989* (Amsterdam: Elsevier)
- [4] Mehta M L 1991 *Random Matrices* (New York: Academic)
- [5] Bohigas O in [3]
- [6] Berry M V 1985 *Proc. R. Soc. A* **400** 229
- [7] Altshuler B L and Shklovskii B I 1986 *JETP* **64** 127
- [8] Berry M V and Tabor M 1977 *Proc. R. Soc. A* **356** 375
- [9] Marcus C M, Rimberg A J, Westervelt R M, Hopkins P F and Gossard A C 1992 *Phys. Rev. Lett.* **69** 509  
Marcus C M, Westervelt R M, Hopkins P F and Gossard A C 1993 *Chaos* **3** 643
- [10] Chang A M, Baranger H U, Pfeiffer L N and West K W 1994 *Phys. Rev. Lett.* **73** 2111
- [11] Fromhold T M, Eaves L, Sheard F W, Foster T J, Leadbeater M L and Main P C 1994 *Physica* **201B** 367;  
1994 *Phys. Rev. Lett.* **72** 2608
- [12] Doron E and Smilansky U 1990 *Phys. Rev. Lett.* **65** 3072; 1991 *Physica* **50D** 367
- [13] Jalabert R A, Baranger H U and Stone A D 1990 *Phys. Rev. Lett.* **65** 2442; 1993 *Chaos* **3** 665
- [14] Jensen R V 1991 *Chaos* **1** 101  
Lin W A, Delos J B and Jensen R V 1993 *Chaos* **3** 655
- [15] Levy L P, Reich D H, Pfeiffer L and West K 1993 *Physica* **189B** 204

- [16] Weiss D, Roukes M L, Menschig A, Grambow P, Klitzing K V and Weimann G 1991 *Phys. Rev. Lett.* **66** 2790  
 Lorke A, Kotthaus J P and Ploog K 1991 *Phys. Rev. B* **44** 3447  
 Weiss D, Richter K, Menschig A, Bergmann R, Schweizer H, von Klitzing K and Weimann G 1993 *Phys. Rev. Lett.* **70** 4118
- [17] Ullmo D, Richter K and Jalabert R A 1995 *Phys. Rev. Lett.* **74** 383
- [18] Gefen, Braun D and Montambaux G 1994 *Phys. Rev. Lett.* **73** 154; 1994 *Ann. Phys.* **3** 467
- [19] Fleischmann R, Geisel T and Ketzmerick R 1992 *Phys. Rev. Lett.* **68** 1367
- [20] Hackenbroich G and von Oppen F *Preprint*
- [21] Kudrolli A, Kidambi V and Sridhar S 1995 *Phys. Rev. Lett.* **75** 822  
 Stein J and Stöckman H-J 1992 *Phys. Rev. Lett.* **68** 2867  
 Sridhar S and Heller E J 1992 *Phys. Rev. A* **46** 1728
- [22] Altland A and Gefen Y 1993 *Phys. Rev. Lett.* **71** 3339
- [23] Altland A and Gefen Y 1995 *Phys. Rev. B* **51** 10671
- [24] Hauge E H 1974 *Transport Phenomena (Lecture Notes in Physics)* ed G Kirczenow and J Marro vol 31 (Berlin: Springer) p 337  
 Cohen E G D 1974 *Colloq. Int. (CNRS)* **236** 269
- [25] Berry M V and Tabor M 1977 *J. Phys. A: Math. Gen.* **10** 371
- [26] Vattay G, Wirzba A and Rosenquist P R 1994 *Phys. Rev. Lett.* **73** 2304  
 Pavloff N and Schmit C 1995 *Phys. Rev. Lett.* **75** 61  
 Primack H, Ussishkin I and Smilansky U, to be published
- [27] Haake F, Kus M and Delande D 1993 *Phys. Rev. Lett.* **71** 2167  
 Kus M, Haake F and Eckhardt B 1993 *Z. Phys. B* **92** 221
- [28] Sieber M, Smilansky U, Creagh S C and Littlejohn R G 1993 *J. Phys. A: Math. Gen.* **26** 6217
- [29] Gutzwiller M C 1967 *J. Math. Phys.* **8** 1979; 1969 **10** 1004; 1970 *J. Math. Phys.* **11** 1791; 1971 *J. Math. Phys.* **12** 343
- [30] Berry M V in [3]
- [31] Argaman N, Imry Y and Smilansky U 1993 *Phys. Rev. B* **47** 4440
- [32] Andreev A V and Altshuler B L 1995 *Phys. Rev. Lett.* **75** 902
- [33] Agam O, Altshuler B L and Andreev A V 1995 *Phys. Rev. Lett.* **75** 4389
- [34] Hannay J H and Ozorio de Almeida A M 1984 *J. Phys. A: Math. Gen.* **17** 3429
- [35] Hall P 1988 *Introduction to the Theory of Coverage Processes* (New York: Wiley); It is our pleasure to thank Robert Adler for bringing [35–37] to our attention and for illuminating discussions on this subject
- [36] Siegel A F 1978 *J. Appl. Prob.* **15** 774
- [37] Siegel A F 1979 *Ann. Prob.* **7** 651
- [38] Ruelle D 1978 *Statistical Mechanics, Thermodynamic Formalism* (Reading, MA: Addison-Wesley)
- [39] Kravtsov V E and Lerner I V 1995 *Phys. Rev. Lett.* **74** 2563
- [40] Messiah A 1970 *Quantum Mechanics* (Amsterdam: North-Holland) p 394
- [41] Averbuch P G 1986 *J. Phys. A: Math. Gen.* **19** 2325. The result for two dimensions is easily obtained with the help of the formulas in this paper. It is our pleasure to thank Eric Akkermans for drawing our attention to this paper and for informative discussions on this problem